



Measuring and testing the ordinal correlation between valued outranking relations

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- **Fitness measures for posteriors** in inverse multiple criteria preference analysis,
- **Fine-tuning meta-heuristics** for multiple criteria based clustering,
- **Comparing multiple criteria rankings** with rules like Kemeny's, Kohler's, the PROMETHEE net flows rule, or, more recently, Dias-Lamboray's prudent leximin rule.



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Kendall's rank correlation τ measure

Let R_1 and R_2 be two binary relations defined on the same finite set X of dimension n .

Let $C = \#\{(x, y) \in X^2 : x \neq y \text{ and } ((x R_1 y) \Leftrightarrow (x R_2 y))\}$ denote the number of concordant non reflexive relational situations we observe.

$$\tau(R_1, R_2) := 2 \times \frac{C}{n(n-1)} - 1$$



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The τ correlation measure – continue

Comment

- *Unanimously concordant relations* (100% equivalent situations) are matched to a correlation index of value **+1.0**,
- *50% concordance* between the relations (50% equivalent and 50% not equivalent situations) is matched to a **zero-valued** correlation index, and
- *Unanimously discordant relations* (100% non equivalent situations) are matched to a correlation index of value: **-1.0**.



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Example

- R_1, R_2 defined on $X = \{a, b, c\}$
- $R_1 = \{(b, c), (c, a)\}$
- $R_2 = \{(a, b), (a, c), (b, c)\}$
- Concordant pairs:
 $\{(b, a), (b, c), (c, b)\}$
- Discordant pairs:
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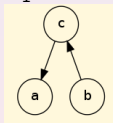
$$\tau(R_1, R_2) = 2 \times \frac{3}{6} - 1 = 0.0$$



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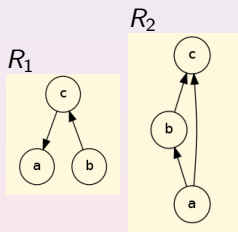


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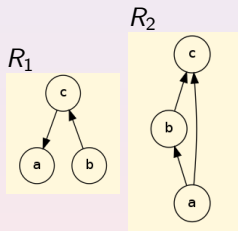


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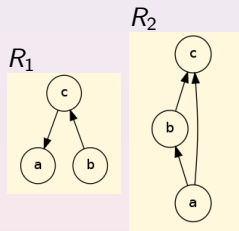


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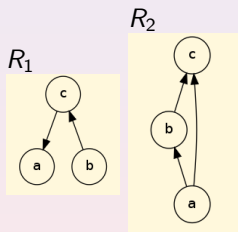


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r -valued relations of order n

- Let R_1 and R_2 be two binary relations defined on the same finite set X of dimension n and characterized via a **bipolar characteristic function** r taking values in the rational interval $[-1.0; 1.0]$.

We call such relations, for short, *r -valued* and *of order n* .

- The r -valuation supports the following semantics:

- $r(xRy) = 1.0$ signifies that the relational situation xRy is totally valid (i.e. more valid than invalid).
- $r(xRy) > 0.0$ signifies that the relational situation xRy is more valid than invalid.
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Logical r -valued operators

$$\begin{aligned}
 r(\neg(x R y)) &= -r(x R y) \\
 r((x R_1 y) \wedge (x R_2 y)) &= \min(r(x R_1 y), r(x R_2 y)), \\
 r((x R_1 y) \vee (x R_2 y)) &= \max(r(x R_1 y), r(x R_2 y)).
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 &r((x R_1 y) \Leftrightarrow (x R_2 y)) \\
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Determinateness of r -valued relations

The **determinateness** of an r -valued relation R of order n , denoted $d(R)$, is defined as follows:

$$d(R) := \frac{\sum_{\substack{x \neq y \\ (x,y) \in X^2}} \text{abs}(r(x R y))}{n(n-1)}$$

Comment

- A *crisp* – a completely 1-valued – relation shows a determinateness degree of 1, whereas
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A useful result

The **equivalence** of two r -valued relational situations verifies the following

Property

Let R_1 and R_2 be any two r -valued relations defined on the same set X . For all (x, y) in X^2 , we have:

$$r((x R_1 y) \Leftrightarrow (x R_2 y)) = \pm \min(\text{abs}(r(x R_1 y)), \text{abs}(r(x R_2 y)))$$



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Comment

- *In the crisp case, following Kendall, we divide the sum of pairwise equivalences by $n(n - 1)$.*
- *If we would proceed this way in the valued case, the resulting measure would integrate a mixture of both the ordinal correlation as well as the actual determinateness of the equivalence observed between the considered r -valued relations.*
- *To factor out both these effects we take, instead, as denominator the maximum possible sum of r -valued equivalences we could potentially observe when both r -valued relations would show completely concordant relational situations.*

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Example

Table : Examples of randomly valued relations

$r(x R_1 y)$	a	b	c
a	–	+0.68	+0.35
b	–0.94	–	+0.80
c	–1.00	+0.36	–

$r(x R_2 y)$	a	b	c
a	–	–0.32	+0.58
b	–0.14	–	+0.75
c	–1.00	+0.08	–



Example

Table : Examples of randomly valued relations

$r(x R_1 y)$	a	b	c
a	–	+0.68	+0.35
b	–0.94	–	+0.80
c	–1.00	+0.36	–
$r(x R_2 y)$	a	b	c
a	–	–0.32	+0.58
b	–0.14	–	+0.75
c	–1.00	+0.08	–



Example – continue

Table : r -valued equivalence between R_1 and R_2

$r(x R_1 y \Leftrightarrow x R_2 y)$	a	b	c
a	–	–0.32	+0.35
b	+0.14	–	+0.75
c	+1.00	+0.08	–

$$\begin{aligned} \tau(R_1, R_2) &= \frac{-0.32 + 0.35 + 0.14 + 0.75 + 1.00 + 0.08}{+0.32 + 0.35 + 0.14 + 0.75 + 1.00 + 0.08} \\ &= \frac{0.200}{0.264} = +0.7575 \end{aligned}$$

$$d(R_1 \Leftrightarrow R_2) = \frac{0.264}{6} = 0.44$$



Example – continue

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$$d(R_1 \Leftrightarrow R_2) = \frac{0.264}{6} = 0.44$$



Properties of the ordinal correlation measure

Let R_1 and R_2 be two r -valued binary relations defined on a same set X :

1. If R_1 and R_2 show a **same**, respectively an **opposite, orientation**, $\tau(R_1, R_2)$ equals $+1.0$, respectively -1.0 , independently of their equivalence determinateness $d((R_1 \Leftrightarrow R_2))$.
2. If $\neg R$ and \mathcal{R} denote resp. the **negation** and the **converse** of relation R , we may notice that:

$$\begin{aligned} \tau(R_1, R_2) &= \tau(R_2, R_1) \\ \tau(\neg R_1, R_2) &= -\tau(R_1, R_2) \\ \tau(\mathcal{R}_1, \mathcal{R}_2) &= \tau(R_1, R_2) \\ \tau(\neg \mathcal{R}_1, \neg \mathcal{R}_2) &= \tau(R_1, R_2) \end{aligned}$$



Properties of the ordinal correlation measure

Let R_1 and R_2 be two r -valued binary relations defined on a same set X :

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1. Motivation

2. Measuring the ordinal correlation

Kendall's approach

Valued equivalence of relational situations

Correlations between r -valued relations

3. Testing the correlation

Randomly r -valued relations

Random weakly complete r -valued relations

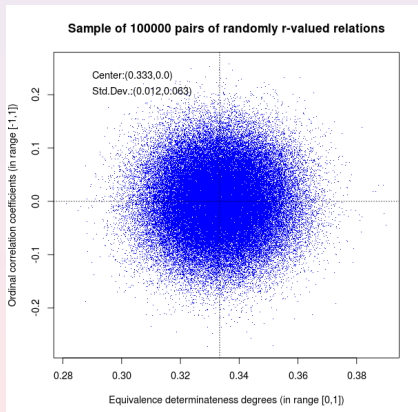
Random r -valued outranking relations

4. Conclusion



Correlation between r -valued relations of order n

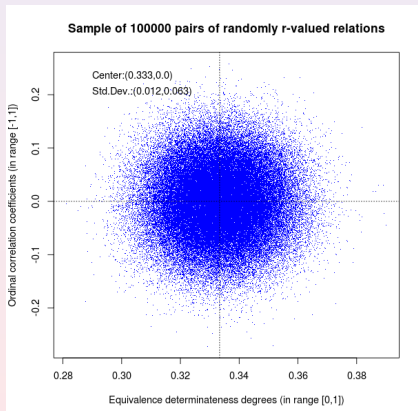
- To each non reflexive pair (x, y) in X are associated two uniform random floats: $r(x R_1 y)$ and $r(x R_2 y)$.
- $r(x R_1 y \Leftrightarrow x R_2 y) \sim \Delta(-1, 1, 0)$, $\mu_e = 0$, $\sigma_e = \sqrt{3/18}$
- $d(x R_1 y \Leftrightarrow x R_2 y) \sim \Delta(0, 1, 0)$: $\mu_d = 1/3$, $\sigma_d = \sqrt{1/18}$.
- $T_n(R_1, R_2) \rightsquigarrow \mathcal{N}\left(\frac{\mu_e}{\mu_d}, \frac{\sigma_e}{\mu_d} \frac{1}{\sqrt{n(n-1)}}\right)$
- $\hat{\mu}_{r_n} \approx 0.0$
- $\hat{\sigma}_{r_n} \approx \frac{3\sqrt{3/18}}{\sqrt{n(n-1)}}$





Correlation between r -valued relations of order n

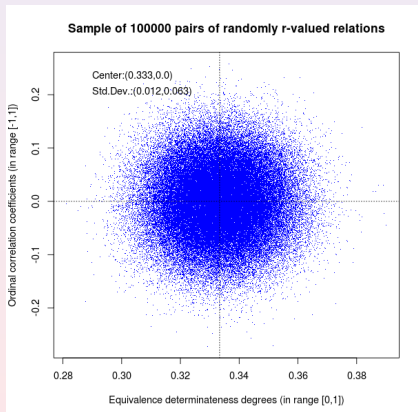
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- $\mathcal{T}_n(R_1, R_2) \rightsquigarrow \mathcal{N}\left(\frac{\mu_e}{\mu_d}, \frac{\sigma_e}{\mu_d} \frac{1}{\sqrt{n(n-1)}}\right)$
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Correlation between r -valued relations of order n

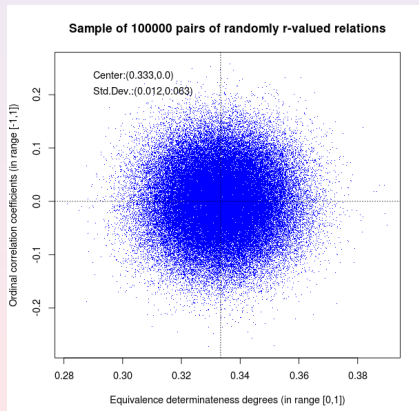
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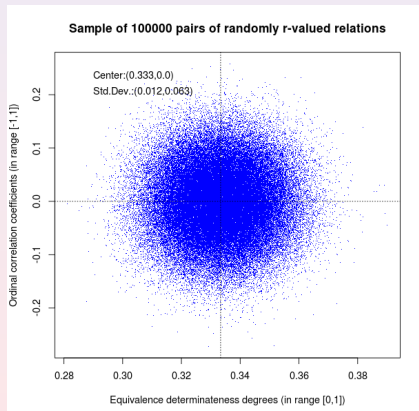
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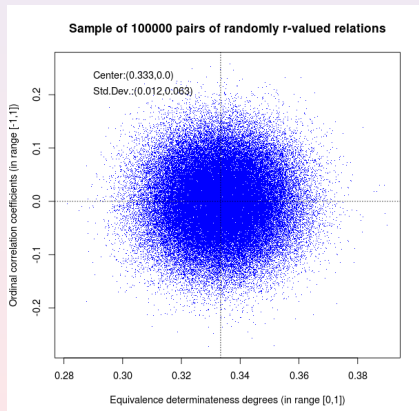


Table : Summary Statistics, for 100000 pairs of randomly r -valued relations

$d(R_1, R_2)$	\bar{d}	$\hat{\sigma}_d$	$\hat{\sigma}_d \sqrt{n(n-1)}$	Conf. 90%	Conf. 99%
$n = 5$	0.3333	0.0527	0.23568	± 0.0866	± 0.1355
$n = 10$	0.3334	0.0249	0.23622	± 0.0406	± 0.0645
$n = 15$	0.3333	0.0162	0.23476	± 0.0266	± 0.0418
$n = 20$	0.3333	0.0121	0.23587	± 0.0202	± 0.0276
$n = 30$	0.3333	0.0080	0.23597	± 0.0132	± 0.0207
$n = 50$	0.3333	0.0048	0.23758	± 0.0078	± 0.0121
$\tau(R_1, R_2)$	$\bar{\tau}$	$\hat{\sigma}_\tau$	$\hat{\sigma}_\tau \sqrt{n(n-1)}$	Conf. 90%	Conf. 99%
$n = 5$	0.0003	0.2731	1.22134	± 0.4500	± 0.6766
$n = 10$	0.0000	0.1289	1.22285	± 0.2181	± 0.3291
$n = 15$	0.0000	0.0842	1.22017	± 0.1386	± 0.2156
$n = 20$	0.0000	0.0621	1.21055	± 0.1035	± 0.1425
$n = 30$	0.0000	0.0414	1.22113	± 0.0681	± 0.1064
$n = 50$	0.0000	0.0247	1.22259	± 0.0406	± 0.0636

$$\sigma_d = \sqrt{1/18} = 0.23570, \quad \frac{\sigma_\tau}{\mu_d} = 3\sqrt{3/18} = 1.22474.$$

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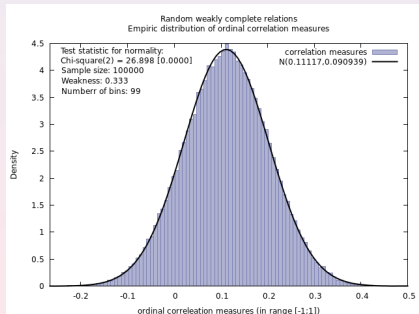
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Random weakly complete r -valued relations

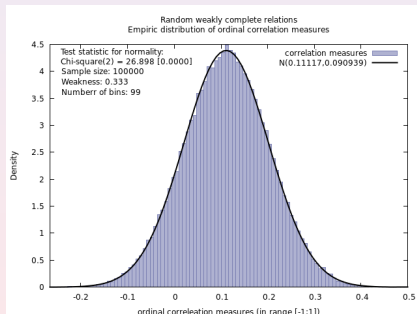
- R is weakly complete if for all $(x, y) \in X$, $r(x R y) < 0$ implies $r(x R y) \geq 0$.
- Each link is, either a double, or a single forward or backward link, with equal probability $1/3$.
- $\hat{\mu}_\tau = +0.111$.
- A weakness degree of 1.0 (resp. 0.0) gives a constant correlation measure of 1.0 (resp. 0.0).





Random weakly complete r -valued relations

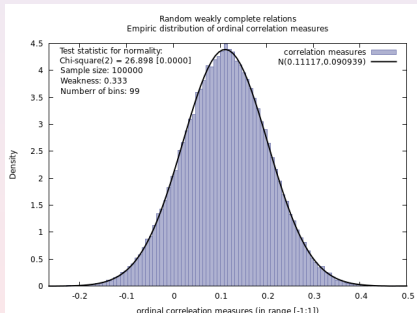
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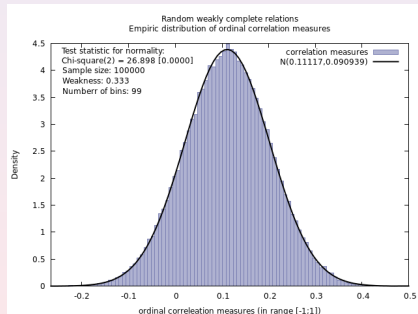


Table : Summary Statistics, for 100000 pairs of random weakly (1/3) complete relations

$d(R_1, R_2)$	$\hat{\mu}_d$	$\hat{\sigma}_d$	$\hat{\sigma}_d \sqrt{n(n-1)}$	Conf. 90%		Conf. 99%	
$n = 5$	0.33344	0.05268	0.23568	0.24920	0.42208	0.20493	0.47489
$n = 10$	0.33316	0.02490	0.23622	0.29262	0.37433	0.27069	0.39861
$n = 15$	0.33316	0.01634	0.23476	0.30646	0.36025	0.29164	0.37578
$n = 20$	0.33320	0.01209	0.23587	0.31342	0.35315	0.30262	0.36486
$n = 30$	0.33316	0.00799	0.23597	0.32006	0.34630	0.31269	0.35406
$n = 50$	0.33318	0.00477	0.23758	0.32537	0.34102	0.32099	0.34558

$\tau(R_1, R_2)$	$\hat{\mu}_\tau$	$\hat{\sigma}_\tau$	$\hat{\sigma}_\tau \sqrt{n(n-1)}$	Conf. 90%		Conf. 99%	
$n = 5$	0.1112	0.3032	1.3560	-0.3981	+0.6039	-0.6559	+0.8229
$n = 10$	0.1113	0.1592	1.5103	-0.1537	+0.3713	-0.3019	+0.5082
$n = 15$	0.1112	0.1138	1.6491	-0.0767	+0.2978	-0.1810	+0.3978
$n = 20$	0.1112	0.0909	1.7720	-0.0395	+0.2604	-0.1231	+0.3399
$n = 30$	0.1116	0.0681	2.0087	-0.0003	+0.2234	-0.0631	+0.2869
$n = 50$	0.1112	0.0484	2.3957	+0.0313	+0.1905	-0.0132	+0.2348

$$\sigma_d = \sqrt{1/18} = 0.23570, \quad \frac{\sigma_\tau}{\mu_d} = 3\sqrt{3/18} = 1.22474.$$

Table : Summary Statistics, for 100000 pairs of random weakly (1/3) complete relations

$d(R_1, R_2)$	$\hat{\mu}_d$	$\hat{\sigma}_d$	$\hat{\sigma}_d \sqrt{n(n-1)}$	Conf. 90%		Conf. 99%	
$n = 5$	0.33344	0.05268	0.23568	0.24920	0.42208	0.20493	0.47489
$n = 10$	0.33316	0.02490	0.23622	0.29262	0.37433	0.27069	0.39861
$n = 15$	0.33316	0.01634	0.23476	0.30646	0.36025	0.29164	0.37578
$n = 20$	0.33320	0.01209	0.23587	0.31342	0.35315	0.30262	0.36486
$n = 30$	0.33316	0.00799	0.23597	0.32006	0.34630	0.31269	0.35406
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$n = 50$	0.1112	0.0484	2.3957	+0.0313	+0.1905	-0.0132	+0.2348

$$\sigma_d = \sqrt{1/18} = 0.23570, \quad \frac{\sigma_e}{\mu_d} = 3\sqrt{3/18} = 1.22474.$$

Table : Summary Statistics, for 100000 pairs of random weakly (1/3) complete relations

$d(R_1, R_2)$	$\hat{\mu}_d$	$\hat{\sigma}_d$	$\hat{\sigma}_d \sqrt{n(n-1)}$	Conf. 90%		Conf. 99%	
$n = 5$	0.33344	0.05268	0.23568	0.24920	0.42208	0.20493	0.47489
$n = 10$	0.33316	0.02490	0.23622	0.29262	0.37433	0.27069	0.39861
$n = 15$	0.33316	0.01634	0.23476	0.30646	0.36025	0.29164	0.37578
$n = 20$	0.33320	0.01209	0.23587	0.31342	0.35315	0.30262	0.36486
$n = 30$	0.33316	0.00799	0.23597	0.32006	0.34630	0.31269	0.35406
$n = 50$	0.33318	0.00477	0.23758	0.32537	0.34102	0.32099	0.34558
$\tau(R_1, R_2)$	$\hat{\mu}_\tau$	$\hat{\sigma}_\tau$	$\hat{\sigma}_\tau \sqrt{n(n-1)}$	Conf. 90%		Conf. 99%	
$n = 5$	0.1112	0.3032	1.3560	-0.3981	+0.6039	-0.6559	+0.8229
$n = 10$	0.1113	0.1592	1.5103	-0.1537	+0.3713	-0.3019	+0.5082
$n = 15$	0.1112	0.1138	1.6491	-0.0767	+0.2978	-0.1810	+0.3978
$n = 20$	0.1112	0.0909	1.7720	-0.0395	+0.2604	-0.1231	+0.3399
$n = 30$	0.1116	0.0681	2.0087	-0.0003	+0.2234	-0.0631	+0.2869
$n = 50$	0.1112	0.0484	2.3957	+0.0313	+0.1905	-0.0132	+0.2348

$$\sigma_d = \sqrt{1/18} = 0.23570, \quad \frac{\sigma_\epsilon}{\mu_d} = 3\sqrt{3/18} = 1.22474.$$



A model of random outranking relations

- Three types of decision actions: *cheap*, *neutral* and *expensive* ones with an equal proportion of 1/3.
- Two types of weighted criteria: *cost* criteria to be *minimized*, and *benefit* criteria to be *maximized*; in the proportions 1/3 respectively 2/3.
- Random performances on each type of criteria, either from an ordinal scale $[0; 10]$, or from a cardinal scale $[0.0; 100.0]$, following a parametric triangular law of mode: 30% performance for cheap, 50% for neutral, and 70% performance for expensive decision actions, with constant probability repartition 0.5 on each side of the respective mode.
- Cost criteria use mostly cardinal scales (3/4), whereas benefit criteria use mostly ordinal scales (2/3).
- The sum of weights of the cost criteria always equals the sum of weights of the benefit criteria.
- On cardinal criteria, both of cost or of benefit type, we observe following constant preference discrimination quantiles: 5% indifferent situations, 90% strict preference situations 90%, and 5% veto situation.
- We call this random model of r -valued relations for short random CB-outranking relations.



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A model of random outranking relations

- Three types of decision actions: *cheap*, *neutral* and *expensive* ones with an equal proportion of $1/3$.
- Two types of weighted criteria: *cost* criteria to be *minimized*, and *benefit* criteria to be *maximized*; in the proportions $1/3$ respectively $2/3$.
- Random performances on each type of criteria, either from an ordinal scale $[0; 10]$, or from a cardinal scale $[0.0; 100.0]$, following a parametric triangular law of mode: 30% performance for cheap, 50% for neutral, and 70% performance for expensive decision actions, with constant probability repartition 0.5 on each side of the respective mode.
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A model of random outranking relations

- Three types of decision actions: *cheap*, *neutral* and *expensive* ones with an equal proportion of $1/3$.
- Two types of weighted criteria: *cost* criteria to be *minimized*, and *benefit* criteria to be *maximized*; in the proportions $1/3$ respectively $2/3$.
- Random performances on each type of criteria, either from an ordinal scale $[0; 10]$, or from a cardinal scale $[0.0; 100.0]$, following a parametric triangular law of mode: 30% performance for cheap, 50% for neutral, and 70% performance for expensive decision actions, with constant probability repartition 0.5 on each side of the respective mode.
- Cost criteria use mostly cardinal scales ($3/4$), whereas benefit criteria use mostly ordinal scales ($2/3$).
- The sum of weights of the cost criteria always equals the sum of weights of the benefit criteria.
- On cardinal criteria, both of cost or of benefit type, we observe following constant preference discrimination quantiles: 5% indifferent situations, 90% strict preference situations 90%, and 5% veto situation.
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A model of random outranking relations

- Three types of decision actions: *cheap*, *neutral* and *expensive* ones with an equal proportion of $1/3$.
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Table : Summary Statistics, for 100000 pairs of random *CB*-outranking relations

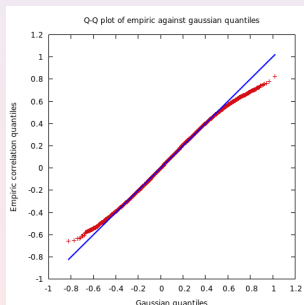
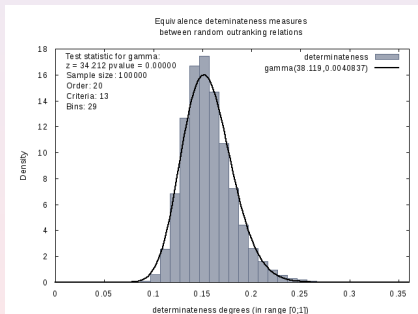
$d(R_1, R_2)$	$\hat{\mu}_d$	$\hat{d}_{50\%}$	$\hat{\sigma}_d$	Conf. 90%		Conf. 99%	
$n = 5, c = 3$	0.3259	0.3250	0.1131	0.1500	0.5255	0.0750	0.6333
$n = 10, c = 7$	0.2207	0.2165	0.0482	0.1494	0.3072	0.1204	0.3681
$n = 15, c = 9$	0.1910	0.1867	0.0362	0.1399	0.2577	0.1196	0.3102
$n = 20, c = 13$	0.1557	0.1527	0.0252	0.1203	0.2013	0.1053	0.2435
$n = 30, c = 21$	0.1372	0.1357	0.0174	0.1120	0.1674	0.1002	0.1989
$\tau(R_1, R_2)$	$\hat{\mu}_\tau$	$\hat{\tau}_{50\%}$	$\hat{\sigma}_\tau$	Conf. 90%		Conf. 99%	
$n = 5, c = 3$	0.0378	0.0345	0.5145	-0.7929	+0.8610	-1.0000	+1.0000
$n = 10, c = 7$	0.0629	0.0644	0.3037	-0.4420	+0.5560	-0.6483	+0.7467
$n = 15, c = 9$	0.0727	0.0761	0.2417	-0.3323	+0.4667	-0.5206	+0.6354
$n = 20, c = 13$	0.0984	0.1017	0.2085	-0.2492	+0.4383	-0.4224	+0.5904
$n = 30, c = 21$	0.1239	0.1272	0.1712	-0.1639	+0.4007	-0.3162	+0.5339

Table : Summary Statistics, for 100000 pairs of random *CB*-outranking relations

$d(R_1, R_2)$	$\hat{\mu}_d$	$\hat{d}_{50\%}$	$\hat{\sigma}_d$	Conf. 90%		Conf. 99%	
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$n = 20, c = 13$	0.0984	0.1017	0.2085	-0.2492	+0.4383	-0.4224	+0.5904
$n = 30, c = 21$	0.1239	0.1272	0.1712	-0.1639	+0.4007	-0.3162	+0.5339



Correlation between pairs of random CB-outrankings



Application

Table : Example CB-outranking relation R_1 ($n = 10$, $c = 7$, $d(R_1) = 0.397$)

R_1	1	2	3	4	5	6	7	8	9	10
1	—	+0.14	+0.43	-0.14	+0.29	+0.14	+0.43	-0.14	± 0.00	+0.43
2	+0.43	—	+0.43	-0.43	+0.43	+0.14	+0.14	+0.43	+0.14	+0.14
3	-0.43	-0.43	—	-0.71	+0.43	+0.00	-0.43	-1.00	-1.00	-0.14
4	+0.14	+0.71	+1.00	—	+0.71	+0.43	+0.71	+0.43	+0.14	+0.57
5	+0.14	-0.43	-0.43	-0.71	—	-0.71	-1.00	+0.14	-1.00	-0.43
6	-0.14	-0.14	+1.00	-0.43	+0.71	—	-0.14	-0.14	+0.14	-0.43
7	+0.14	+0.14	+0.43	-0.43	+1.00	+0.14	—	+0.14	+0.43	+0.29
8	+0.43	-0.14	+1.00	-0.43	+0.43	+0.14	-0.14	—	+0.43	-0.14
9	+1.00	-0.14	+1.00	-0.14	+1.00	+0.43	+0.14	-0.14	—	+0.14
10	-0.43	-0.14	+0.43	+0.14	+0.43	+0.43	+0.29	+0.14	-0.14	—

Assessing different ranking rules:

1. Kemeny ranking: $R_2 = [4, 2, 7, 8, 9, 1, 10, 6, 3, 5]$, $\nu(R_1, R_2) = +0.888$.

2. Net flow score: $MF = [4, 9, 2, 7, 8, 10, 1, 6, 3, 5]$, $\nu(R_1, MF) = +0.776$.

3. Fisher ranking: $R_3 = [4, 2, 8, 10, 9, 6, 1, 7, 3, 5]$, $\nu(R_1, R_3) = +0.776$.

4. Ranked sets (main): $R_4 = [4, 2, 8, 9, 1, 7, 10, 6, 3, 5]$, $\nu(R_1, R_4) = +0.872$.

Application

Table : Example CB-outranking relation R_1 ($n = 10$, $c = 7$, $d(R_1) = 0.397$)

R_1	1	2	3	4	5	6	7	8	9	10
1	—	+0.14	+0.43	-0.14	+0.29	+0.14	+0.43	-0.14	± 0.00	+0.43
2	+0.43	—	+0.43	-0.43	+0.43	+0.14	+0.14	+0.43	+0.14	+0.14
3	-0.43	-0.43	—	-0.71	+0.43	+0.00	-0.43	-1.00	-1.00	-0.14
4	+0.14	+0.71	+1.00	—	+0.71	+0.43	+0.71	+0.43	+0.14	+0.57
5	+0.14	-0.43	-0.43	-0.71	—	-0.71	-1.00	+0.14	-1.00	-0.43
6	-0.14	-0.14	+1.00	-0.43	+0.71	—	-0.14	-0.14	+0.14	-0.43
7	+0.14	+0.14	+0.43	-0.43	+1.00	+0.14	—	+0.14	+0.43	+0.29
8	+0.43	-0.14	+1.00	-0.43	+0.43	+0.14	-0.14	—	+0.43	-0.14
9	+1.00	-0.14	+1.00	-0.14	+1.00	+0.43	+0.14	-0.14	—	+0.14
10	-0.43	-0.14	+0.43	+0.14	+0.43	+0.43	+0.29	+0.14	-0.14	—

Assessing different ranking rules:

1. Kemeny ranking: $Ke = [4, 2, 7, 8, 9, 1, 10, 6, 3, 5]$, $\tau(R_1, Ke) = +0.888$,
2. Net flow scores: $NF = [4, 9, 2, 7, 8, 10, 1, 6, 3, 5]$, $\tau(R_1, NF) = +0.776$,
3. Kohler ranking: $Ko = [4, 2, 8, 10, 9, 6, 1, 7, 3, 5]$, $\tau(R_1, Ko) = +0.776$,
4. Ranked pairs (bwin): $Rp = [4, 2, 8, 9, 1, 7, 10, 6, 3, 5]$, $\tau(R_1, Rp) = +0.872$.

Application

Table : Example CB-outranking relation R_1 ($n = 10$, $c = 7$, $d(R_1) = 0.397$)

R_1	1	2	3	4	5	6	7	8	9	10
1	—	+0.14	+0.43	-0.14	+0.29	+0.14	+0.43	-0.14	± 0.00	+0.43
2	+0.43	—	+0.43	-0.43	+0.43	+0.14	+0.14	+0.43	+0.14	+0.14
3	-0.43	-0.43	—	-0.71	+0.43	+0.00	-0.43	-1.00	-1.00	-0.14
4	+0.14	+0.71	+1.00	—	+0.71	+0.43	+0.71	+0.43	+0.14	+0.57
5	+0.14	-0.43	-0.43	-0.71	—	-0.71	-1.00	+0.14	-1.00	-0.43
6	-0.14	-0.14	+1.00	-0.43	+0.71	—	-0.14	-0.14	+0.14	-0.43
7	+0.14	+0.14	+0.43	-0.43	+1.00	+0.14	—	+0.14	+0.43	+0.29
8	+0.43	-0.14	+1.00	-0.43	+0.43	+0.14	-0.14	—	+0.43	-0.14
9	+1.00	-0.14	+1.00	-0.14	+1.00	+0.43	+0.14	-0.14	—	+0.14
10	-0.43	-0.14	+0.43	+0.14	+0.43	+0.43	+0.29	+0.14	-0.14	—

Assessing different ranking rules:

1. Kemeny ranking: $Ke = [4, 2, 7, 8, 9, 1, 10, 6, 3, 5]$, $\tau(R_1, Ke) = +0.888$,
2. Net flow scores: $Nf = [4, 9, 2, 7, 8, 10, 1, 6, 3, 5]$, $\tau(R_1, Nf) = +0.776$,
3. Kohler ranking: $Ko = [4, 2, 8, 10, 9, 6, 1, 7, 3, 5]$, $\tau(R_1, Ko) = +0.776$,
4. Ranked pairs (leximin): $Rp = [4, 2, 8, 9, 1, 7, 10, 6, 3, 5]$, $\tau(R_1, Rp) = +0.872$.

Application

Table : Example CB-outranking relation R_1 ($n = 10$, $c = 7$, $d(R_1) = 0.397$)

R_1	1	2	3	4	5	6	7	8	9	10
1	–	+0.14	+0.43	–0.14	+0.29	+0.14	+0.43	–0.14	± 0.00	+0.43
2	+0.43	–	+0.43	–0.43	+0.43	+0.14	+0.14	+0.43	+0.14	+0.14
3	–0.43	–0.43	–	–0.71	+0.43	+0.00	–0.43	–1.00	–1.00	–0.14
4	+0.14	+0.71	+1.00	–	+0.71	+0.43	+0.71	+0.43	+0.14	+0.57
5	+0.14	–0.43	–0.43	–0.71	–	–0.71	–1.00	+0.14	–1.00	–0.43
6	–0.14	–0.14	+1.00	–0.43	+0.71	–	–0.14	–0.14	+0.14	–0.43
7	+0.14	+0.14	+0.43	–0.43	+1.00	+0.14	–	+0.14	+0.43	+0.29
8	+0.43	–0.14	+1.00	–0.43	+0.43	+0.14	–0.14	–	+0.43	–0.14
9	+1.00	–0.14	+1.00	–0.14	+1.00	+0.43	+0.14	–0.14	–	+0.14
10	–0.43	–0.14	+0.43	+0.14	+0.43	+0.43	+0.29	+0.14	–0.14	–

Assessing different ranking rules:

1. Kemeny ranking: $Ke = [4, 2, 7, 8, 9, 1, 10, 6, 3, 5]$, $\tau(R_1, Ke) = +0.888$,
2. Net flow scores: $Nf = [4, 9, 2, 7, 8, 10, 1, 6, 3, 5]$, $\tau(R_1, Nf) = +0.776$,
3. Kohler ranking: $Ko = [4, 2, 8, 10, 9, 6, 1, 7, 3, 5]$, $\tau(R_1, Ko) = +0.776$,
4. Ranked pairs (leximin): $Rp = [4, 2, 8, 9, 1, 7, 10, 6, 3, 5]$, $\tau(R_1, Rp) = +0.872$.

Application

Table : Example CB-outranking relation R_1 ($n = 10$, $c = 7$, $d(R_1) = 0.397$)

R_1	1	2	3	4	5	6	7	8	9	10
1	—	+0.14	+0.43	-0.14	+0.29	+0.14	+0.43	-0.14	± 0.00	+0.43
2	+0.43	—	+0.43	-0.43	+0.43	+0.14	+0.14	+0.43	+0.14	+0.14
3	-0.43	-0.43	—	-0.71	+0.43	+0.00	-0.43	-1.00	-1.00	-0.14
4	+0.14	+0.71	+1.00	—	+0.71	+0.43	+0.71	+0.43	+0.14	+0.57
5	+0.14	-0.43	-0.43	-0.71	—	-0.71	-1.00	+0.14	-1.00	-0.43
6	-0.14	-0.14	+1.00	-0.43	+0.71	—	-0.14	-0.14	+0.14	-0.43
7	+0.14	+0.14	+0.43	-0.43	+1.00	+0.14	—	+0.14	+0.43	+0.29
8	+0.43	-0.14	+1.00	-0.43	+0.43	+0.14	-0.14	—	+0.43	-0.14
9	+1.00	-0.14	+1.00	-0.14	+1.00	+0.43	+0.14	-0.14	—	+0.14
10	-0.43	-0.14	+0.43	+0.14	+0.43	+0.43	+0.29	+0.14	-0.14	—

Assessing different ranking rules:

1. Kemeny ranking: $Ke = [4, 2, 7, 8, 9, 1, 10, 6, 3, 5]$, $\tau(R_1, Ke) = +0.888$,
2. Net flow scores: $Nf = [4, 9, 2, 7, 8, 10, 1, 6, 3, 5]$, $\tau(R_1, Nf) = +0.776$,
3. Kohler ranking: $Ko = [4, 2, 8, 10, 9, 6, 1, 7, 3, 5]$, $\tau(R_1, Ko) = +0.776$,
4. Ranked pairs (leximin): $Rp = [4, 2, 8, 9, 1, 7, 10, 6, 3, 5]$, $\tau(R_1, Rp) = +0.872$.

Application

Table : Example CB-outranking relation R_1 ($n = 10$, $c = 7$, $d(R_1) = 0.397$)

R_1	1	2	3	4	5	6	7	8	9	10
1	—	+0.14	+0.43	-0.14	+0.29	+0.14	+0.43	-0.14	± 0.00	+0.43
2	+0.43	—	+0.43	-0.43	+0.43	+0.14	+0.14	+0.43	+0.14	+0.14
3	-0.43	-0.43	—	-0.71	+0.43	+0.00	-0.43	-1.00	-1.00	-0.14
4	+0.14	+0.71	+1.00	—	+0.71	+0.43	+0.71	+0.43	+0.14	+0.57
5	+0.14	-0.43	-0.43	-0.71	—	-0.71	-1.00	+0.14	-1.00	-0.43
6	-0.14	-0.14	+1.00	-0.43	+0.71	—	-0.14	-0.14	+0.14	-0.43
7	+0.14	+0.14	+0.43	-0.43	+1.00	+0.14	—	+0.14	+0.43	+0.29
8	+0.43	-0.14	+1.00	-0.43	+0.43	+0.14	-0.14	—	+0.43	-0.14
9	+1.00	-0.14	+1.00	-0.14	+1.00	+0.43	+0.14	-0.14	—	+0.14
10	-0.43	-0.14	+0.43	+0.14	+0.43	+0.43	+0.29	+0.14	-0.14	—

Assessing different ranking rules:

1. Kemeny ranking: $Ke = [4, 2, 7, 8, 9, 1, 10, 6, 3, 5]$, $\tau(R_1, Ke) = +0.888$,
2. Net flow scores: $Nf = [4, 9, 2, 7, 8, 10, 1, 6, 3, 5]$, $\tau(R_1, Nf) = +0.776$,
3. Kohler ranking: $Ko = [4, 2, 8, 10, 9, 6, 1, 7, 3, 5]$, $\tau(R_1, Ko) = +0.776$,
4. Ranked pairs (leximin): $Rp = [4, 2, 8, 9, 1, 7, 10, 6, 3, 5]$, $\tau(R_1, Rp) = +0.872$.

Application

Table : Example CB-outranking relation R_1 ($n = 10$, $c = 7$, $d(R_1) = 0.397$)

R_1	1	2	3	4	5	6	7	8	9	10
1	—	+0.14	+0.43	-0.14	+0.29	+0.14	+0.43	-0.14	± 0.00	+0.43
2	+0.43	—	+0.43	-0.43	+0.43	+0.14	+0.14	+0.43	+0.14	+0.14
3	-0.43	-0.43	—	-0.71	+0.43	+0.00	-0.43	-1.00	-1.00	-0.14
4	+0.14	+0.71	+1.00	—	+0.71	+0.43	+0.71	+0.43	+0.14	+0.57
5	+0.14	-0.43	-0.43	-0.71	—	-0.71	-1.00	+0.14	-1.00	-0.43
6	-0.14	-0.14	+1.00	-0.43	+0.71	—	-0.14	-0.14	+0.14	-0.43
7	+0.14	+0.14	+0.43	-0.43	+1.00	+0.14	—	+0.14	+0.43	+0.29
8	+0.43	-0.14	+1.00	-0.43	+0.43	+0.14	-0.14	—	+0.43	-0.14
9	+1.00	-0.14	+1.00	-0.14	+1.00	+0.43	+0.14	-0.14	—	+0.14
10	-0.43	-0.14	+0.43	+0.14	+0.43	+0.43	+0.29	+0.14	-0.14	—

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Conclusion

- We have consistently generalized Kendall's rank correlation measure τ to r -valued binary relations via a corresponding r -valued logical equivalence measure.
- The so extended ordinal correlation measure, besides remaining identical to Kendall's measure in the case of completely determined linear orders, shows interesting properties like its independence with the actual determinateness degree of the r -valued equivalence.
- Empirical confidence intervals for different models of random r -valued relations, like weakly complete and, more particularly, r -valued outranking relations are elaborated.

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